Often in physics and math we define things in the most concrete and intuitive way we can.
But sometimes allowing a more abstract definition allows us to extend the applicability of the idea at hand.

For example we first defined trigonometric functions $\sin\theta, \cos\theta, \tan\theta$, etc. in terms of real triangles that we could physically construct. In fact one way to "calculate" $\cos\theta$ is to create a right triangle with one of its other angles $= \theta$, then measure the length of the adjacent side and divide by the length of the hypotenuse.

This definition and its utility is rooted in our immediate perception of geometry.

But then what should we make of $\cos(i\theta)$ where $i = \sqrt{-1}$? Certainly we can't build a triangle with one side of imaginary length, much less measure it.

However... if we allow ourselves to define $\cos x$ by its Taylor series, i.e. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$
then not only can we evaluate $\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots$
but also $\cos(i\theta) = 1 - \frac{1}{2!}(i\theta)^2 + \frac{1}{4!}(i\theta)^4 + \cdots$

This also allows us to see the connection between $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} = \cos(i\theta)$!

Thus far our discussion of rotations and Lorentz transformations has heavily focussed on concrete representations like coordinates on a space. After all $dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_\mu dx^\mu$ is defined by how it acts on coordinate differentials. From this basic starting point we can build more general vectors, dual vectors, scalars, and higher rank tensors.

But is there something more?

Strangely, in answering this question we also learn a method for comparing continuous groups. Recall that for finite discrete groups we can just compare their multiplication tables to see if they are the same, e.g.

$$\begin{array}{c|cc} & E & O \\ \hline E & E & O \\ O & O & E \end{array} \quad = \quad \begin{array}{c|cc} & +1 & -1 \\ \hline +1 & +1 & -1 \\ -1 & -1 & +1 \end{array}$$

However for continuous groups there are an ∞-number of elements so constructing multiplication tables is hopeless.

But we do know that for the groups we are concerned with there seem to be only a handful of distinct "types" of transformations, and everything else is built out of these, e.g. rotations in 3D ~ R_{xy}, R_{yz}, R_{zx}.

Brainstorm: What might we compare to see if 2 continuous groups are the same?

- # of free parameters ✓
- abelian vs. non-abelian ✓
- dimensionality of matrices ✗
- real or complex ✗

Let's formalize this...

Fortunately most of the groups we use are examples of Lie groups. This includes the Standard Model groups $SO(1,3)$, $SU(3)$, $SU(2)$, $U(1)$ as well as purely spatial rotations $SO(N)$.

Technically: A Lie group forms a manifold with a differentiable structure.

What is important about this is that if we know how to do something infinitesimally then we can (almost) figure everything else out.

For example if we know a matrix that rotates in the yz-plane by 1°, i.e. $R_{yz}(1^\circ)$, then we could build any integer rotation in yz by compounding these, e.g. $R_{yz}(45^\circ) = [R_{yz}(1^\circ)]^{45}$.

Of course if we don't want to limit ourselves to integer transformations then we need to start with an infinitesimal rotation $R_{yz}(\delta\theta)$ where $\delta\theta$ is infinitesimal.

The hard part is figuring out how to compound infinitesimal transformations to get a finite transformation. Fortunately Lie groups come equipped with a way to do this.

Formalizing this: A general element of a Lie group can be written as $A = e^{i g_A v^A}$ where g_A is said to "generate" A and v^A is a parameter.

Note: i) If A is an $N \times N$ matrix, then so is g_A.

ii) The combination $g_A v^A$ allows us to combine transformations in different directions, e.g. $g_A v^A = g_{R_{yz}} \theta + g_{R_{zx}} \phi + g_{R_{xy}} \psi$ for $SO(3)$.

This is good because we know that $R_{yz}(\theta) R_{zx}(\phi) R_{xy}(\psi)$ must combine to give a single rotation!

Example: $R_{yz}(\theta) = \underbrace{e^{i g_{R_{yz}} \theta}}_{\text{exponential of a matrix}} = I + i g_{R_{yz}} \theta - \frac{1}{2!} g^2_{R_{yz}} \theta^2 + \frac{1}{3!} (i g_{R_{yz}} \theta)^3 + \cdots$

WTF!?

On vectors this is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 - \frac{1}{2!}\theta^2 + \cdots & \theta - \frac{1}{3!}\theta^3 + \cdots \\ 0 & -\theta + \frac{1}{3!}\theta^3 + \cdots & 1 - \frac{1}{2!}\theta^2 + \cdots \end{pmatrix} \Rightarrow$ $g_{R_{yz}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

Note: $g^2_{R_{yz}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$g^3_{R_{yz}} = g_{R_{yz}}$, etc.

similarly $g_{R_{zx}} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

$g_{R_{xy}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So we can think about $SO(3)$ in terms of its "infinitesimal generators" $g_{R_{yz}}, g_{R_{zx}}, g_{R_{xy}}$.
The 3 matrices are not unique, in fact if you start with these 3 you can get another set of 3 by (you might have guessed) rotating them (all in the same way). It is very akin to choosing coordinates.

Pro-tip: The generators actually form a basis in the tangent space of the group manifold at the origin, so the analogy to coordinates is quite deep.

Now our discussion so far has been very coordinate dependent. That is we started with 3x3 matrices, inspired by how rotations act on coordinates (and hence vectors like dx^{μ}).

To approach things in a coordinate independent manner we consider the "algebra" of the generators: $[g_{R_{yz}}, g_{R_{zx}}] = g_{R_{yz}} g_{R_{zx}} - g_{R_{zx}} g_{R_{yz}} = i g_{R_{xy}}$

$$\Downarrow$$

(letting $i,j,k = 1,2,3$) $\quad \underbrace{[g_i, g_j] = i\epsilon^{ijk} g_k}_{\text{The "Lie algebra" of SO(3)}}$ $\quad \epsilon^{123} = \epsilon^{312} = \epsilon^{231} = +1$, $\epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1$, $\epsilon^{iij} = 0$

(this is coordinate independent)

A Lie group can be characterized in the vicinity of the identity by the Lie algebra of its generators.

So far in our discussion of rotations in 3D we have encountered scalars, vectors and tensors.
These are 0, 3 and higher dimensional representations of the rotation group.

What about a 2D representation of 3D rotations?

We would need 2×2 matrices satisfying $[g_i, g_j] = i\epsilon^{ijk} g_k$ where $\underbrace{i,j,k=1,2,3}_{3D}$

$[g_1, g_2] = i g_3$
$[g_2, g_3] = i g_1$
$[g_3, g_1] = i g_2$

These work: $g_{R_{yz}} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x$ $\quad g_{R_{zx}} = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} = \frac{1}{2}\sigma_y$ $\quad g_{R_{xy}} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \frac{1}{2}\sigma_z$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices

Now we can build: $R_{yz}(\theta) = e^{i g_{R_{yz}} \theta} = \underbrace{\begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}}$ and similarly for R_{zx} and R_{xy}.

satisfy $U^\dagger U = \mathbb{I}$ and $\det U = +1$ } SU(2) which act on complex 2-component spinors χ.

Often we write $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ $\qquad$ Note: We will not use spin indices in this class, so we will rely on matrix manipulations.

So $SO(3) \sim SU(2)$, at least near the identity (which is all the Lie algebra knows about).

Globally however there is a difference: $\quad SO(3) \quad R_x(2\pi) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} = \mathbb{I}$
$\quad SU(2) \quad R_x(2\pi) = \begin{pmatrix} -1&0 \\ 0&-1 \end{pmatrix} = -\mathbb{I}$

} SU(2) is called the double-cover of SO(3)

of course $R_x(4\pi) = \mathbb{I}$ for both!

There is a certain sense in which spinors and SU(2) probes geometry more deeply than coordinates, scalars, vectors, SO(3), etc.
By "probe more deeply" I mean they contain more information. Sometimes people say that spinors know about the square root of the geometry.
In fact if we consider the anti-commutator of the Pauli matrices we find: $\underbrace{\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times2}}_{\text{A Clifford algebra.}}$

Example: $\sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} + \begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix} = \begin{pmatrix} i&0 \\ 0&-i \end{pmatrix} + \begin{pmatrix} -i&0 \\ 0&i \end{pmatrix} = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$ as expected since $\delta_{xy} = 0$

$\sigma_y\sigma_y + \sigma_y\sigma_y = 2\sigma_y\sigma_y = 2\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} = 2\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$ as expected since $\delta_{yy} = 1$

It might seem silly, but recall that $\delta_{ij} = \begin{pmatrix} 1 && \\ &1& \\ &&1 \end{pmatrix}$ is the metric of $\mathbb{R}^3$. This will come in handy later.

Another illustration of this is a lesson from QM: If we only have integer spin states at our disposal, $\{0, 1, 2, \dots\}$ then by combining spins we can only ever build more integer spin states.
However if we allow ½ integer spin states, then we can build ½ or whole integer states just using ½ spin states, e.g. $\frac{1}{2} - \frac{1}{2} = 0$, $\frac{1}{2} + \frac{1}{2} = 1$.